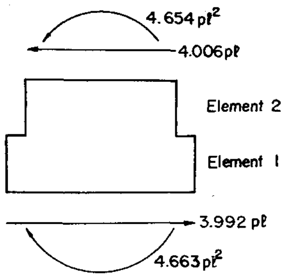

Fig. 1 Equilibrium of the node.

rors); i.e., there are no stress resultant discontinuities at the nodes. This is true for the structural system of the example or for any other system.

Parenthetically, it is worthwhile to mention that the stiffness matrix and the generalized forces of the structural elements of the example are exact since the third-order polynomial is the exact deflection pattern for a straight flexural beam element with constant (EI). This is not the case when approximate deflection patterns are used as, for example, in the work on shells of revolution by the Massachusetts Institute of Technology group.

Consider the node joining elements 1 and 2 of the example (Fig. 1) with the stress resultants taken from columns 1 and 2 of the last equation of the Note. The discontinuities in the bending moments ($0.009pl^2$) and in shear forces ($0.014pl$) are exclusively due to computational errors. Theoretically there should not be such discontinuities since this situation corresponds precisely to the satisfaction of the node equilibrium conditions. The writer worked out this example by a standard matrix structural analysis FORTRAN program. It was found that the stress resultants, from each side of the nodes, agreed with the exact solution with five significant figures.

In conclusion, contrary to the assertion of the Note that "although this example is too simple to draw general conclusions from . . . ," it can be conclusively asserted that the method is an exact one even for shells of revolution. For this type of structural system the only approximation is due to the use of approximate deformation functions for the shell elements. However this does not cause any stress resultant discontinuities at the nodes.

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[1] Stricklin, J. A., "Computation of Stress Resultants from the Element Stiffness Matrices," *AIAA Journal*, Vol. 4, No. 6, June 1966, pp. 1095–1096.

[2] Filho, F. V., "Matrix Analysis of Plane Rigid Frames," *Transactions of the American Society of Civil Engineers*, Vol. 126-II, 1961, pp. 214–227.

[3] Filho, F. V., "Discussion on Generalized Displacements in Structural Analysis, by J. L. Meek," *Journal of the Structural Division, Proceedings of the American Society of Civil Engineers*, Vol. 88, ST6, 1963, pp. 303–316.

Reply to F. V. Filho

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PROFESSOR Filho has studied my technical note in detail and retained more significant figures in his calculations. Although the equations used in my technical note[1] and Ref. 2 of Professor Filho's comments are the same, our objectives were quite different. In particular my objective was to obtain the stress resultants after computation of the generalized displacements. My assertion about not being able to draw general conclusions from a simple example has been verified by more recent studies on shells of revolution.[2] It was found that my method yields good results when the geometry of the shell is accurately represented but yields poor results when the shell is represented by conical segments. Based on this more recent data we can say that the method is theoretically correct but sensitive to geometric approximation.

The statement "since the third order polynomial is the exact deflection pattern for a straight flexural beam element with constant (EI)," is not correct. For example, the exact deflection pattern for a uniform beam under a uniform loading is a fourth-order polynomial. In summary, we used the same equations to reach different objectives. Otherwise, I believe the limitation as stated in Ref. 1 should remain.

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[2] Navaratna, D. R., "Computation of Stress Resultants in Finite Element Analysis," *AIAA Journal*, Vol. 4, No. 11, Nov. 1966, pp. 2058–2060.

Torque on a Satellite in a General Gravitational Field

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SCHLEGEL[1] has properly pointed out that I was careless (he was kind enough not to use the word) in an earlier derivation of the gravitational torque on a satellite in a non-inverse square field.† He provides a correct development and, in particular, gives forms for the body components of torque in the neighborhood of the oblate earth. His results are expressed partly in terms of body orientation parametrized by Euler angles, though portions of the equations are simplified considerably by his leaving them in terms of the direction cosines, which he denotes by $m_{\alpha\beta}$ $(\alpha,\beta = 1,2,3)$.

In this Note, I would like to give an alternative derivation which is somewhat more compact and is more straightforward, in my opinion, in regard to the development of the torque components from the potential. It leads to a compact matrix form for the result that is more general in several particulars; a form that also is extremely convenient for the numerical computation of these torques for the purposes of digital simulation.

Denote by $U(\mathbf{R})$ the specific potential at the center of mass of a material system due to the gravitational field in which it is immersed, $\mathbf{R}$ locating this center of mass with respect to the center of the earth or other central body. Denote by $U(\mathbf{R} + \rho)$ the specific potential at a nearby point. Integrating the latter over the material system and using a well-known expansion from vector analysis, valid when the gradient operator is expressed in rectangular coordinates, the spe-

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† Several others have pointed out to me the fallacy of my original development, and I wish to take this opportunity to express to them my genuine appreciation.

cific potential of the entire system can be expressed as

$$U = U(\mathbf{R}) + \frac{1}{2m}\,\mathrm{grad}\cdot\left(\frac{1}{2}\,trI\mathbf{E} - \mathbf{I}\right)\cdot\mathrm{grad}\,U + \ldots \quad (1)$$

Here $\mathbf{E}$ is the unit dyadic, $\mathbf{I}$ is the inertia dyadic, and the implied derivatives are evaluated at $\boldsymbol{\varrho} = 0$.

For the moment, we associate an orbit with the material system purely for the purpose of defining a convenient reference frame, though this does not impose a restriction on the generality of the result. Let $\{\boldsymbol{\xi}_\alpha\}$ be an orthonormal vector basis, along the outward geocentric vertical through the center of mass, $\boldsymbol{\xi}_2$ normal to the plane of the orbit such that $\boldsymbol{\xi}_1$ is in the forward sense. Let‡ y^α be spatial coordinates along $\boldsymbol{\xi}_\alpha$ and write grad $= \boldsymbol{\xi}_\alpha \partial/\partial y^\alpha$. Let $\{\mathbf{X}_\alpha\}$ be an orthonormal basis embedded in the material system, with respect to which the inertia tensor has the components $I_{\alpha\beta}$. Suppose that the body frame is related to the $\boldsymbol{\xi}$ frame by $\mathbf{X}_\alpha = \Theta_{\alpha\beta}\boldsymbol{\xi}_\beta$. It is a detail to reduce Eq. (1) to

$$U = U(\mathbf{R}) + \frac{1}{2m}\left(\frac{1}{2}\,trI\delta_{\lambda\mu} - I_{\lambda\mu}\right)\Theta_{\lambda\alpha}\Theta_{\mu\beta}U_{\alpha\beta} + \ldots \quad (2)$$

where $U_{\alpha\beta} \equiv \partial^2 U/\partial y^\alpha \partial y^\beta$ evaluated at $y^\alpha = 0$.

To relate changes in the potential function to torques on the body, one may recognize that if a virtual change in body orientation is made, the work done on the body in the virtual displacement is the change in total potential. But it is also the work done by the external torques. The former can be written $\delta U = (\partial U/\partial\Theta_{\alpha\beta})\delta\Theta_{\alpha\beta}$ (not summed) for the variation of a direction cosine, while the latter is $\delta U = L_\alpha \delta\theta_\alpha$ with L_α the body components of torque and $\delta\theta_\alpha$ the virtual rotation angles about body axes. (Note that these angles are considered infinitesimal, so there is no resolution problem.) Moreover, $\delta\Theta_{\alpha\beta} = \epsilon_{\alpha\mu\lambda}\Theta_{\mu\beta}\delta\theta_\lambda$. This can be shown by a direct argument, but it is simpler to recognize that it is just an infinitesimal incremental form of the Euler-Lagrange-Poisson kinematical equations for rotating frames. Combining these results, we have

$$L_\alpha = \epsilon_{\alpha\lambda\rho}\Theta_{\rho\mu}(\partial U/\partial\Theta)_{\lambda\mu} \quad (3)$$

Putting Eq. (2) into Eq. (3),

$$L_\alpha = \epsilon_{\alpha\rho\lambda}I_{\lambda\xi}\Theta_{\rho\mu}\Theta_{\xi\beta}U_{\mu\beta} \quad (4)$$

Note that $U_{\mu\beta}$ was formed using rectangular coordinates y^α. If a transformation to a curvelinear set $\bar{y}^\alpha$ is made, it is known that

$$U_{\mu\beta} = U_{,\rho\sigma}\,(\partial\bar{y}^\rho/\partial y^\mu)(\partial\bar{y}^\sigma/\partial y^\beta) \quad (5)$$

where $U_{,\rho\sigma}$ is the covariant derivative with respect to the $\bar{y}$ coordinates. To take maximum advantage of the compaction possible with a matrix formulation, introduce matrices

$$T = [T_{\sigma\beta}] = [\partial\bar{y}^\sigma/\partial y^\beta] \qquad U, = [U_{,\sigma\rho}] \quad (6)$$

Then $U^* \equiv [U_{\mu\beta}] = T^T U, T$ and

$$L_\alpha = \epsilon_{\alpha\rho\lambda}(I\Theta U^*\Theta^T)_{\lambda\rho} \quad (7)$$

As a special case when body axes are principal axes, the symmetry of $\Theta U^*\Theta^T$ implies

$$L_1 = (I_3 - I_2)(\Theta U^*\Theta^T)_{32} \quad (8)$$

with two similar equations by cyclic permutation of indices.§

‡ Greek lower case indices have the range 1,2,3 throughout. The summation convention is followed.

§ Pengelley[2] gave a very similar result, though in a specialized form that assumed the derivatives in U^* necessarily to be with respect to rectangular coordinates. The formula was not applied in that work.

As an example, suppose that

$$U = (k/r)[1 + (C/r^2)(3\sin^2\phi - 1)] \quad (9)$$

as it closely is for the oblate earth, where $C = \frac{1}{2}R_E{}^2C_{2,0}$, R_E is the equatorial radius, ϕ is the geocentric latitude, and r is the magnitude of vector $\mathbf{R}$ introduced previously. The calculation of U^* is straightforward, if a bit tedious. Letting ζ denote the angle from the easterly direction to the satellite forward direction $\boldsymbol{\xi}_1$, measured positively about $\boldsymbol{\xi}_3$, one obtains

$$U_{11} = -\frac{k}{r^3}\left\{1 + \frac{3C}{r^2}\,[(5\sin^2\phi - 1) - 2\cos^2\phi\sin^2\zeta]\right\} \quad (10a)$$

$$U_{12} = U_{21} = \frac{3kC}{r^5}\cos^2\phi\sin 2\zeta \quad (10b)$$

$$U_{13} = U_{31} = -\frac{12kC}{r^5}\sin 2\phi\sin\zeta \quad (10c)$$

$$U_{22} = -\frac{k}{r^3}\left\{1 + \frac{3C}{r^2}\,[(5\sin^2\phi - 1) - 2\cos^2\phi\cos^2\zeta]\right\} \quad (10d)$$

$$U_{23} = U_{32} = -\frac{12kC}{r^5}\sin 2\phi\cos\zeta \quad (10e)$$

$$U_{33} = \frac{2k}{r^3}\left\{1 + \frac{6C}{r^2}(3\sin^2\phi - 1)\right\} \quad (10f)$$

Actually, because of the symmetry of the dominant part of the field about $\boldsymbol{\xi}_3$, it is convenient to decompose U^* as

$$U^* = -\frac{k}{r^3}\left[1 + \frac{3C}{r^2}(5\sin^2\phi - 1)\right]E + V \quad (11)$$

where E is the 3×3 unit matrix and

$$V = \frac{3k}{r^3}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} + \frac{3kC}{r^5}M \quad (12)$$

with

$$M = \begin{bmatrix}2\cos^2\phi\sin^2\zeta & \cos^2\phi\sin 2\zeta & -4\sin 2\phi\sin\zeta\\ \cos^2\phi\sin 2\zeta & 2\cos^2\phi\cos^2\zeta & -4\sin 2\phi\cos\zeta\\ -4\sin 2\phi\sin\zeta & -4\sin 2\phi\cos\zeta & (17\sin^2\phi - 5)\end{bmatrix} \quad (13)$$

The portion of U^* involving E, when substituted into Eq. (7), gives a null term, and one is left with

$$L_\alpha = \epsilon_{\alpha\rho\lambda}(I\Theta V\Theta^T)_{\lambda\rho} \quad (14)$$

To compare these results with Schlegel's, we can put $\zeta = 0$ so that our ξ frame corresponds to his triad $\mathbf{j}$, $\mathbf{i}$, $-\mathbf{k}$ (in that order). Recalling that $J_2 = -C_{2,0}$ in the standard notation for the earth's potential, and assuming with him that body axes are principal axes of inertia, we find straightforwardly that

$$(\Theta V\Theta^T)_{32} = \frac{3k}{r^3}\Theta_{33}\Theta_{23} - \frac{3kR_E{}^2J_2}{2r^5}[\Theta_{31}\Theta_{32}\Theta_{33}]M\begin{bmatrix}\Theta_{21}\\ \Theta_{22}\\ \Theta_{23}\end{bmatrix}$$

or

$$L_1 = \frac{3k}{r^3}(I_3 - I_2)\Theta_{23}\Theta_{33} - \frac{3kR_E{}^2J_2}{2r^5}(I_3 - I_2)[2\theta_{22}\theta_{32}\cos^2\phi - 4(\Theta_{22}\Theta_{33} + \Theta_{23}\Theta_{32})\sin 2\phi + \Theta_{23}\Theta_{33}(17\sin^2\phi - 5)] \quad (15)$$

With allowance for notational differences, this is precisely his

T_y. (Note that his $\sin 2\theta \cos\phi$ is better restored to direction cosine form before comparing.) Similarly, it is found that $L_2 =$ his T_x, and $L_3 =$ his $-T_z$, as should be the case because of different axis orientation conventions.

In summary, Eq. (7) is a convenient matrix form for the gravitational torque in a completely general field. It reduces to Schlegel's under the conditions he assumes, but is somewhat more general in that it does not imply that body axes are principal axes, and it allows the body orientation to be measured from a frame with one more degree of rotational freedom. Furthermore, the body orientation is not tied to any specific parametrization of the direction cosine matrix.

I fully agree that the magnitude of the torque correction from oblateness is very small, but I cannot concur with Schlegel that it yet has been established that it "can be safely ignored save in high precision studies." Although I suspect he is correct, it should be recognized that the oblateness effect will appear not only as an "external torque" in the dynamical equations, but also as a parametrix excitation. Under these conditions it is conceivable that it results in changes in stability characteristic or response amplitude to a degree quite unanticipated from its numerical magnitude. I believe that further studies are required to determine whether this kind of behavior can arise in situations which are of practical interest.

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[2] Pengelley, C. D., "Gravitational Torque on a Small Rigid Body in an Arbitrary Field," *ARS Journal*, Vol. 32, 1962, pp. 420–422.

Comment on "Large Deflection of Rectangular Sandwich Plates"

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THIS Comment concerns the statement of the boundary conditions presented by the authors[1] in their Eq. (3) and rewritten below in a different form:

$$\text{at } \xi = \pm 1 \quad U = V = 0 \qquad W = \alpha = \beta = \partial W/\partial \xi = 0$$

$$\text{at } \eta = \pm 1 \quad U = V = 0 \qquad W = \alpha = \beta = \partial W/\partial \eta = 0$$

The equations involving U and V represent possible statements of the necessary two boundary conditions for the inplane system. The equations involving W, α, β, and $\partial W/\partial \xi$ ($\partial W/\partial \eta$) would then represent four boundary conditions for the bending. However, plate bending is described by a sixth-order set of equations, so that there can be just three boundary conditions on each edge. The extra and incorrect boundary conditions are the following:

$$\text{at } \xi = \pm 1 \qquad \partial W/\partial \xi = 0$$

$$\text{at } \eta = \pm 1 \qquad \partial W/\partial \eta = 0$$

The stress-strain-displacement relations for transverse shear are

$$S_x/G_c h = \gamma_{xz} = \alpha + \theta\, \partial W/\partial \xi$$

$$S_y/Q_c h = \gamma_{yz} = \beta + \theta\, \partial W/\partial \eta$$

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where S_x and S_y are the transverse shear forces, and γ_{xz} and γ_{yz} are the transverse shear strains. These equations show that requiring both α and $\partial W/\partial \xi$ to vanish forces zero transverse shear strain at the boundaries $\xi = \pm 1$. Since the edge transverse shear strains are not zero for a sandwich plate, the center deflection will actually be larger than reported by the authors.

Reference

[1] Kan, H. and Huang, J., "Large Deflection of Rectangular Sandwich Plates," *AIAA Journal*, Vol. 5, No. 9, Sept. 1967, pp. 1706–1708.

Reply by Author to C. V. Smith

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SINCE we can always combine α with $\partial W/\partial \xi$ and β with $\partial W/\partial \eta$ to obtain transverse shear strains, boundary conditions (3) of our Note[3] are not independent. They are correct conditions, since the solutions of our Note are given for a rectangular sandwich plate with rigidly clamped edges. The conditions characterizing a rigidly clamped edge parallel to the coordinate-axes are zero deflection and zero slope of the middle surface along the edge and zero rotation of the cross section making up the boundary. These requirements certainly force the transverse shear strain to zero at the boundaries. It can be easily seen by considering the deflection of a cantilever sandwich beam due to deformation associated with the shear stress as shown in Fig. 1. Element A, on the neutral axis, which is originally rectangular, changes into a rhombus, but element B near the clamped end remains unchanged. Hence we can simply assume that the transverse shear strain equals to zero at the restrained end.

In the early works of Hoff,[1,2] similar boundary conditions have been used for the bending of a cantilever sandwich beam and bending of rectangular sandwich plate with edges clamped. The transverse shear strains vanish at the restrained boundaries for both cases.

For the sake of simplicity, we can always relax the boundary conditions to some extent by letting the edge slopes of the middle surface of the plate be different from zero. Naturally this assumption will lead to slightly larger center deflection.

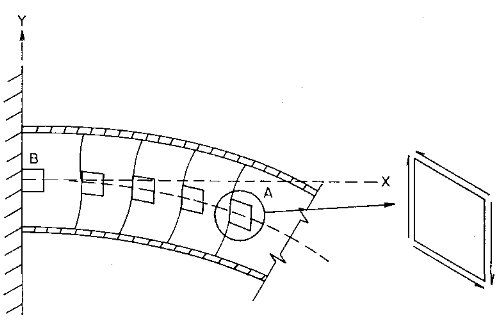

Fig. 1 Deflection of cantilever sandwich beam.

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[1] Hoff, N. J., *The Analysis of Structure*, Wiley, New York, 1956, pp. 180–193; also Hoff, N. J. and Mantner, S. E., "Bending and Buckling of Sandwich Beams," *The Journal of the Aeronautical Sciences*, Vol. 15, No. 12, 1948, p. 707.

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